The number of such inequalities may be found from the coefficient of $c_{1i}c_{2j}c_{3k}c_{4l}$ in the iterated series. The first subscripts of the ϕ's in the first term of the squared factor admit 24 permutations, while the second term may be formed from it in four ways, each of the operators K_1K_2 retaining one of the ϕ's associated with it in the first term and in the opposite position. The 96 results so indicated reduce however to 48 by reason of the permutability of the terms of the squared factor. There are then 48 inequalities of precisely the same type as the one in question; but there are others of the same general type corresponding to any two permutations of the subscripts 1, 2, 3, 4, in the terms of the squared factor, making in all $\binom{24}{2}$ or 276 results, exclusive of the useless one $0 \geqq 0$ which is obtained when the same permutation is used twice. Of these, 24 are reducible inequalities of the type

$$(J_{15}'J_{26}''J_{37}'''J_{48}^{\text{iv}} - J_{15}'J_{26}''J_{38}'''J_{47}^{\text{iv}} - J_{15}'J_{26}''J_{47}'''J_{38}^{\text{iv}} + J_{15}'J_{26}''J_{48}'''J_{37}^{\text{iv}})_{\kappa_1\kappa_2\kappa_1\kappa_2}$$
$$= (J_{15}'J_{26}''{}_{\kappa_1\kappa_1}) \times (J_{37}'''J_{48}^{\text{iv}} - J_{38}'''J_{47}^{\text{iv}} - J_{47}'''J_{38}^{\text{iv}} + J_{48}'''J_{37}^{\text{iv}})_{\kappa_2\kappa_2} \geqq 0,$$

in which also each factor of the left member is separately $\geqq 0$. The other 204 are irreducible inequalities not isomorphic with the proposed formula, and falling into five classes which may be indicated by the permutation of first subscripts $(abcd)$ which would be present in the second term of

$$K_1(\phi_1\phi_2)K_2(\phi_3\phi_4) - K_1(\phi_a\phi_b)K_2(\phi_c\phi_d)$$

(second subscripts being omitted) for a typical case of each class. That is, 96 correspond to the permutation (1342), 48 to (1432), 24 to (3412), 24 to (3421), and 12 to (2143).

2761 [1919, 124]. Proposed by W. W. DENTON, University of Michigan.

Find the lengths of the side of an equilateral triangle whose vertices are at given distances a, b, c from a given point.

I. SOLUTION BY C. E. MANGE, Junior, Washington University.

To construct the triangle.

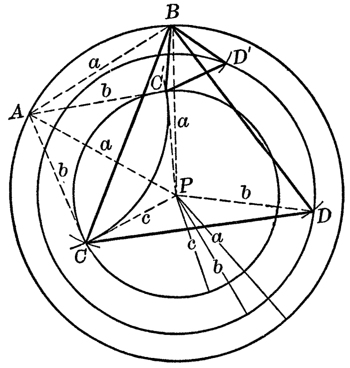
With given point, P, as a center, describe circles with radii a, b, and c, respectively. Assume $a > b > c$. Construct chord AB in circle (a) equal to a, and with A as a center describe an arc with radius b intersecting circle (c) in C and C'. With B as a center describe an arc of radius BC intersecting circle (b) in D.[1] Draw BC, CD, and DB. The triangle BCD is equilateral and fulfils the required conditions. (Similarly for the triangle $BC'D'$.)

Proof: Draw AC, AP, AB, BP, CP, and DP.

Now $AB = BP = PA$, by construction; therefore, $\triangle ABP$ is equilateral, and $\angle PBA = 60°$.

Since $\triangle ACB = \triangle BPD$, $\angle ABC = \angle PBD$; and hence $\angle ABC + \angle CBP = 60° = \angle CBD$. And since $CB = BD$, by construction, CBD is equilateral.

Area of



$$ACP = A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)},$$

$$\angle BAP = 60°, \qquad \text{and} \qquad \angle PAC = \text{arc} \sin \frac{2A}{ba}.$$

In triangle BAC,

$$BC = \sqrt{b^2 + a^2 - 2ba \cos\left[60° + \text{arc} \sin \frac{2A}{ba}\right]}$$
$$= \sqrt{b^2 + a^2 - 2ba\left[\frac{1}{2} \cdot \frac{b^2 + a^2 - c^2}{2ba} - \frac{\sqrt{3}}{2} \cdot \frac{2A}{ba}\right]}$$
$$= \sqrt{\frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}A}.$$
$$BC' = \sqrt{\frac{a^2 + b^2 + c^2}{2} - 2\sqrt{3}A}.$$

[1] D is that intersection which lies on the opposite side of BP from A, so that the four lines from B are in the order BA, BC, BP and BD. D' is the intersection determined in the same way when an arc of radius BC' is described.—EDITORS.

II. Solution by the Proposer.

Consider first the case in which the plane of the desired triangle passes through the given point, and the problem is confined to two dimensions. Let P be the given point, $a \geqq b \geqq c \geqq 0$ the given distances, and e the length of the side of the required triangle. Let the angles between the lines drawn to the vertices be A, B, C, so that $A + B + C = 2\pi$ [2] and, therefore, one has

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2 \cos A \cos B \cos C.$$

From this and the law of cosines, $e^2 = a^2 + b^2 - 2ab \cos C$, etc., comes the equation,

$$e^4 - (a^2 + b^2 + c^2)e^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0,$$

and the values e^2,

$$e^2 = \tfrac{1}{2}(a^2 + b^2 + c^2 \pm \sqrt{\Delta_0}),$$

where

$$\Delta_0 = 3(a + b + c)(a + b - c)(c + a - b)(b + c - a).$$

The values of e^2 are real and unequal, if, and only if, the relation $b + c > a$ is satisfied; they are real and equal if, and only if, the relation $b + c = a$ is satisfied. Moreover, the values of e^2 are never negative; this may be shown by using the relation

$$a^4 + b^4 + c^4 \geqq a^2b^2 + b^2c^2 + c^2a^2.$$

Therefore, there are two, one, or no equilateral triangles having their vertices at the given distances a, b, c, from a given point lying in their plane, according as these distances themselves may be taken as the lengths of the sides of a non-degenerate triangle, a degenerate triangle, or no triangle, respectively.

It can be shown that the given point always lies outside or at the vertex of one found triangle, and that when a^2 is between $b^2 + bc + c^2$ and $(b + c)^2$, it lies outside both triangles. (It is, therefore, not true, as stated in G. R. Perkins's *Plane and Solid Geometry*, New York, 1860, p. 231, that the minus sign must be used before the radical in the above formula when the given point lies outside the triangle.)

In the general case, some variable parameter beside the given lengths must be introduced: let this be the distance p from the given point P to the plane of the desired triangle. The projections of a, b, c, on this plane are

$$(1) \qquad a' = \sqrt{a^2 - p^2}, \qquad b' = \sqrt{b^2 - p^2}, \qquad c' = \sqrt{c^2 - p^2}.$$

The length of the side of an equilateral triangle lying in this plane and having vertices at the distances a', b', c', from the foot of this perpendicular is the length sought, and may therefore be obtained by putting a', b', c', in place of a, b, c, viz.,

$$e^2 = \tfrac{1}{2}(a^2 + b^2 + c^2 - 3p^2 \pm \sqrt{\Delta_p}),$$

where

$$\Delta_p = 9p^4 - 6p^2(a^2 + b^2 + c^2) + \Delta_0 = (3p^2 - \alpha)(3p^2 - \beta),$$

$$\alpha = a^2 + b^2 + c^2 - \sqrt{2}\sqrt{(a^2 + b^2 + c^2)^2 - \Delta_0} = a^2 + b^2 + c^2 - \sqrt{2}\sqrt{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2},$$

and

$$\beta = a^2 + b^2 + c^2 + \sqrt{2}\sqrt{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}.$$

These values of e^2 are real (for real values of p^2), if, and only if, p^2 lies in one of the intervals, $p^2 \leqq \tfrac{1}{3}\alpha$, $\tfrac{1}{3}\beta \leqq p^2$.

For the first interval, e^2 is never negative, but p^2 is negative except when the condition $b + c \geqq a$ is satisfied. For the second interval, both values of e^2 are always negative. These facts have been used in proving the following statements:

For each value of p subject to the condition $0 \leqq p^2 < \tfrac{1}{3}\alpha$, two equilateral triangles are found, provided the given distances a, b, c, may be taken as the lengths of the sides of a triangle.

(S) When the conditions $p^2 = \tfrac{1}{3}\alpha$ and $b + c \geqq a$ are both satisfied, just one triangle is found.

In all other cases no triangle is found.

(D) One of the triangles found degenerates into a point if, and only if, the given distances are equal.

The cases in which the perpendicular from the given point to the triangle plane falls

(1) outside one triangle and inside the other;

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[2] If the given point lies outside of the given triangle we shall have $B + C - A = 0$, but the equations of cosines still hold.—Editors.

(2) on a side of one triangle and outside the other, but (2) (S) at a vertex if there is only one solution; and

(3) outside both triangles, also (3) (S) outside, if there is only one solution:
correspond respectively to the conditions

$$p^2 \lesseqqgtr \frac{b^2c^2 - (b^2 + c^2 - a^2)^2}{2a^2 - (b^2 + c^2)} = f(a^*bc).$$

The case in which the perpendicular falls

(4) on the side of one triangle produced,
is a special case under case (1) and corresponds to the conditions

$$f(a^*bc) > p^2 = \frac{a^2c^2 - (a^2 + c^2 - b^2)^2}{2b^2 - (a^2 + c^2)} = f(b^*ac).$$

These formulas are indeterminate if, and only if, one of the triangles found is degenerate.

$f(a^*bc)$ and $f(b^*ac)$ are equal only when a and b are equal, and they then reduce to $\frac{1}{3}\alpha$; so that the property (2) (S) mentioned above may be regarded as the result of combining any two of the properties (2), (S), and (4).

Only the following eight combinations of the above properties are possible,

(1), (1) (4), (1) (D), (2), (2) (4) (S), (3), (3) (S), (S) (D).

The general conditions which are mentioned above have been derived by "projection" from the corresponding conditions when p is zero, that is, by making the substitutions (1). For example, the conditions that P falls on a side or a side produced are respectively $e = b + c$, $e = b - c$, which are equivalent to $a^2 = b^2 + bc + c^2$, $a^2 = b^2 - bc + c^2$. In the latter condition, when the relation, $a \geqq b \geqq c \geqq 0$, is to be preserved, care must be taken to interchange a and b. The value of a which makes P fall on a side produced is then determined by the equation $a^2 - ac + c^2 = b^2$. Substituting a' for a, etc., in this equation gives the corresponding condition for three dimensions, $p^2 = f(b^*ac)$.

III. Historical Notes by R. C. Archibald, Brown University.

This problem has been frequently discussed in books, pamphlets, and periodicals for more than a century. In 1803 L. N. M. Carnot gave indications of synthetic and analytic solutions of the following more general problem (*Géométrie de position*, Paris, 1803, pp. 381–382, 389–390): "Connoisant les trois angles d'un triangle, et les distances de leurs trois sommets à un point donné dans le même plan, trouver les trois côtés de ce triangle." Carnot points out that the method of solving this problem may be applied to solve the following example concerning the pyramid: "Connoissant tous les angles que font deux à deux les six arêtes d'une pyramide triangulaire, et les distances de ses quatre sommets à un point quelconque de l'espace, trouver toutes les dimensions de cette pyramide." Among many discussions of Carnot's problem in the plane, the following may be mentioned: By R. Götting, *Einen Punkt zu bestimmen, dessen Entfernung von drei gegebenen Punkten sich wie drei gegebene gerade Linien verhalten.* Progr. Torgau, 1888. 30 pp. + 1 plate—By Combier, "Note de géométrie," *Journal de mathématiques élémentaires*, vol. 3, 1879, pp. 120–126 (trigonometric discussion)—By A. H. Curtis, M. Jenkins, and J. McDowell, *Mathematical Questions with their Solutions from the 'Educational Times,'* vol. 44, 1886, p. 110; vol. 45, 1886, pp. 68–70.

The particular case for an equilateral triangle, with its vertices on three concentric circles of given radii, was considered in Gabriel Lamé's valuable *Examen des différentes méthodes employées pour résondre les problèmes de géométrie* (Paris, 1818, pp. 81–82), in G. Ritt's *Problèmes d'application de l'algèbre à la géométrie avec les solutions développées* (Paris, 1836, pp. 17–20), and in *Mathematical Questions with their Solutions from the 'Educational Times,'* (a) by W. S. McCay and W. S. Burnside, vol. 10, 1868, pp. 98–99; (b) by W. J. C. Miller, R. F. Davis, S. A. Renshaw, etc., vol. 26, 1876, pp. 24–28.

In *Nouvelles Annales de Mathématiques*, vol. 3, 1844, p. 376, Prouhet proposed the following problem: "Trois circonférences étant tracées sur un même plan, on propose de trouver sur ces circonférences, *en ne faisant usage que du compas*, trois points qui soient les sommets d'un triangle équilatéral." Solution by Breton (de Champ) is given in vol. 9, 1850, pp. 299–304.

Also solved by R. D. Bohannan, W. F. Cheney, Jr., William Herberg, H. Halperin, H. L. Olson, A. Pelletier, Elijah Swift, and C. C. Yen.